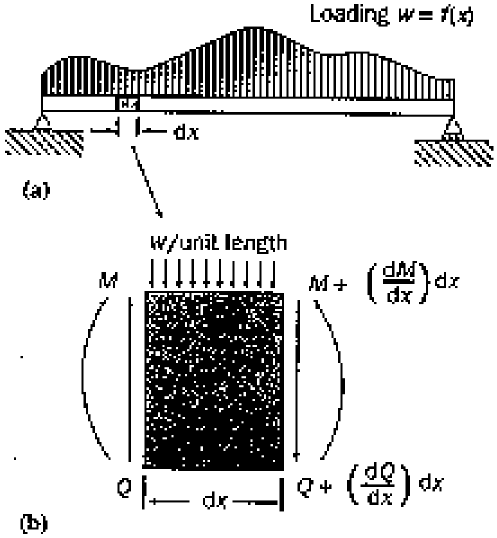
BENDING STRESS

Shear force and bending moment

In general, a beam can be loaded in a way that varies arbitrarily along its length. Consider the case in which we have a varying distributed load w. This will give rise to shear force and bending moments in the beam, for which we can generate mathematical relations. Consider the free body diagram of a section of beam shown. The shear forces and moments at the ends of the section are defined in terms of the rates of change $\frac{dQ}{dx}$ and $\frac{dM}{dx}$.

Fig. 6.6



Force equilibrium gives relationships for the shear force. Taking the downward direction as positive, vertical equilibrium gives

$$Q + \frac{dQ}{dx}dx + wdx - Q = 0$$

from which

$$w = -\frac{dQ}{dx} \qquad (1).$$

Rearranging and integrating,

$$dQ = -wdx \Rightarrow \int_1^2 dQ = \int_1^2 -wdx \Rightarrow Q_2 - Q_1 = -\int_1^2 wdx \qquad (2).$$

Here the labels 1 and 2 refer to points on the beam. The final relation means that the change in shear force between the two points is obtained by integrating the distributed load along the beam.

For this case we take the same beam element and do moment equilibrium. Taking moments about the centre of the element so that w has no effect gives the total anticlockwise moment

$$M + \frac{dM}{dx}dx - M - \left(Q + \frac{dQ}{dx}dx \right)\frac{dx}{2} - Q\frac{dx}{2} = 0$$

where the $\frac{dx}{2}$ terms arise since that is the distance from the centre to the end of the element. Simplifying,

$$\frac{dM}{dx}dx - Q\frac{dx}{2} - \frac{dQ}{dx}\frac{dx^2}{2} - Q\frac{dx}{2} = 0.$$

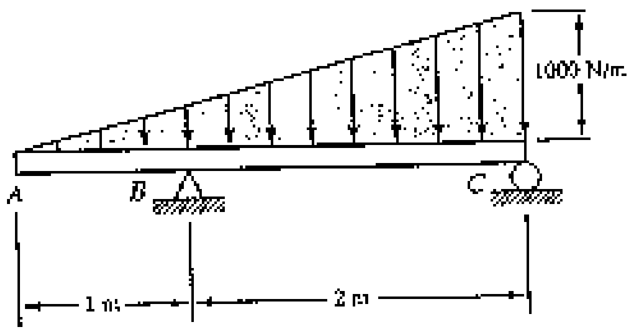
Since dx^2 is second order we can neglect the term in which it appears and the result is

$$\frac{dM}{dx}dx - Qdx = 0 \Rightarrow \frac{dM}{dx} = Q \qquad (3).$$

The above result is in differential form. The corresponding result in integral form is obtained by integration:

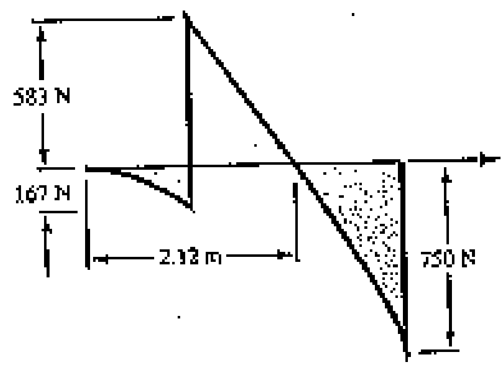
$$M_2 - M_1 = \int_1^2 Qdx \qquad (4).$$

Find (a) shear force and (b)bending moment for this beam.

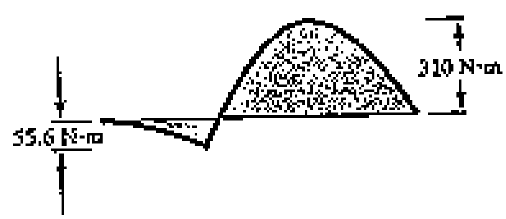


Ans.

(a)



(b) Shear



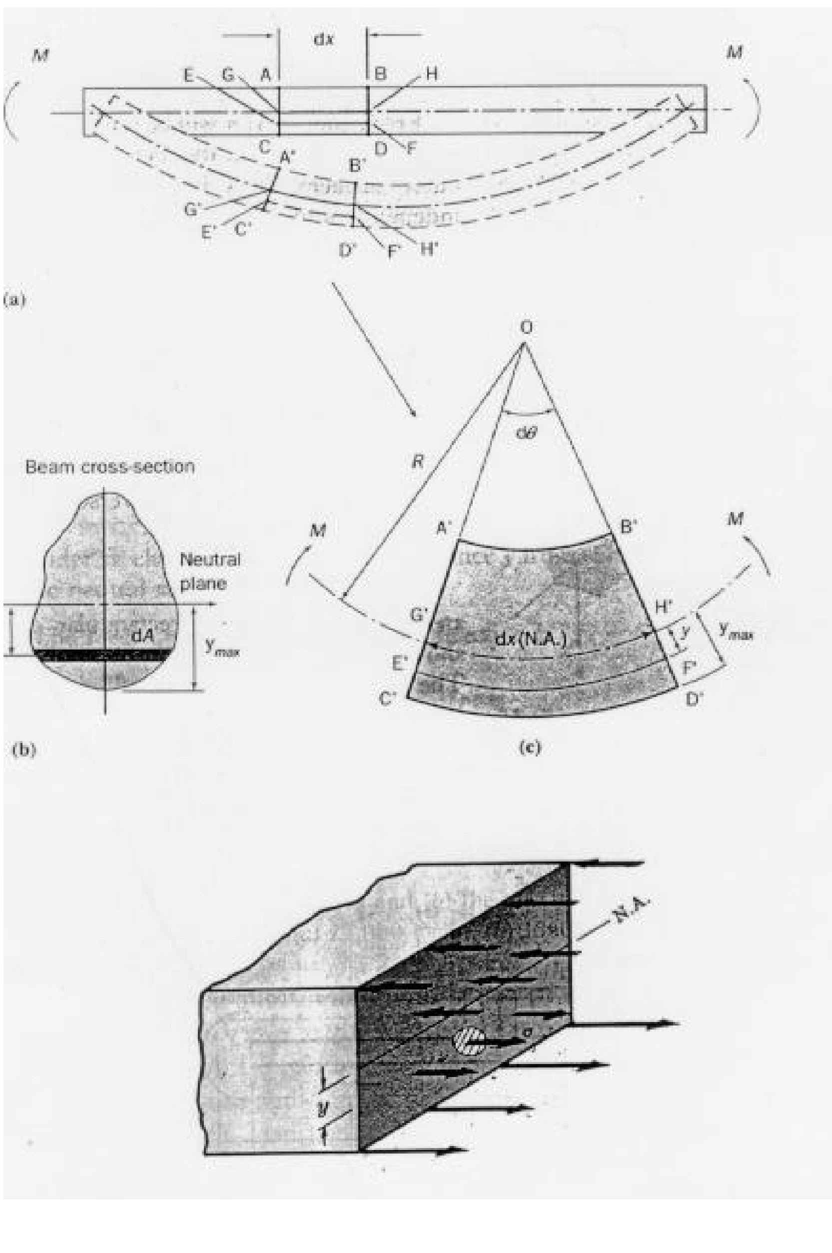
(c) Bending Moment

Bending stress

Consider the section of beam illustrated. Let us assume that the section ABCD is deformed into a circular arc, such that the top surface becomes compressed to the shorter length A'B' and the bottom surface CD stretched to the longer length C'D'. It follows that there is a horizontal plane in the beam that does not change in length; this is known as the neutral surface and denoted GH in the diagram. EF is any plane at a distance y from the neutral surface. Since the deformed section subtends the angle $d\theta$, by definition of the angle in terms of radius r and arc length s "$s = rd\theta$", the deformed length E'F'is given by

$$E'F' = (R + y)d\theta.$$

We can use this relationship to derive an expression for the strain at any point y. Using the definition of strain ε



$$\varepsilon = \frac{\text{change in length}}{\text{original length}} = \frac{(R+y)d\theta - Rd\theta}{Rd\theta}.$$

This follows since the original length is the length of the neutral surface, GH = G'H' = Rdθ. Simplifying the above equation gives

$$\varepsilon = \frac{y}{R} \qquad (5).$$

Note that y is positive below the neutral surface, where the material is stretched and the strain positive. Above the neutral surface, y is negative, giving negative strain. The stresses follow the same pattern, as shown at the bottom of the figure above.

Consider these stresses σ acting horizontally on the exposed and of the element. Suppose this end has an area A. The total force on the element is given by integrating the stress over its area. From equilibrium, we know that the total force is zero and we may write

$$\int_A \sigma dA = 0.$$

Since the stress is only along the x direction, we can use the simple form of Hooke's Law, $\sigma = E\varepsilon$.

Then, equation (5) gives

$$\sigma = \frac{Ey}{R} \qquad (6).$$

Using this in the integral above gives

$$\int_A \frac{y}{R} dA = 0$$

which implies that

$$\int_A y dA = 0 \qquad (7).$$

Recall the definition of the y co-ordinate of the centroid of a surface

$$\bar{y} = \frac{1}{A}\int_A y dA.$$

Equation (7) now implies that $\bar{y} = 0$. Since y is the distance from the centroid, we must conclude that the neutral surface passes through the centroid. This is the first important result for stresses in beams.

The above result was obtained by equilibrium. We may gain further insight by taking moments for the section. The moment generated by the stress field σ must equal the applied moment, which in this case is the bending moment. For a strip of the section of area dA a distance y from the neutral surface, the moment dM generated by the stress in the strip about the neutral surface is (total force in the strip)× y, which is

$$dM = \sigma dA \times y = \sigma y dA.$$

Integrating over the area A then gives

$$M = \int_A \sigma y dA$$

where M is now the bending moment. Now use equation (6):

$$M = \int_A \frac{Ey^2}{R} dA .$$

Taking the constants outside the integral,

$$M = \frac{E}{R} \int_A y^2 dA .$$

The integral above is recognisable as the second moment of area about the neutral surface, I. The relation above can be rewritten as

$$M = \frac{EI}{R} \qquad (8).$$

Now from (6), $\frac{E}{R} = \frac{\sigma}{y}$. Then, (8) becomes

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R} \qquad (9).$$

Equation (9) is known as .. For any position along a beam, the left- and right-hand terms are constant, showing that the stress σ is proportional to y, and is zero on the neutral surface. The greatest tension and compression occur at the beam surfaces.

Suppose we have a bar of rectangular cross-section subject to symmetrical three-point bend and wish to find the stress. The stress is given by equation (9) as

$$\sigma = \frac{My}{I} .$$

We need to find both M and I. Since the ends are simply supported, the bending moment is zero there. It changes linearly and reaches a maximum at the central point load, where the stress is also greatest. The value of M there is $\frac{FL}{4}$. The stresses on this central plane are given by

$$\sigma = \frac{FLy}{4I}.$$

Recall that y is the distance from the centre, and is positive downwards. Therefore, the stress on the bottom surface is positive – tensile - and on the top surface negative – compressive. Suppose that the rectangular section is of width b and height d. The value of I about the neutral axis is (Week 6)

$$I = \frac{bd^3}{12}.$$

The expression for the stress is then

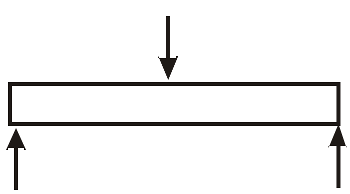
$$\sigma = \frac{12FLy}{4bd^3} = \frac{3FLy}{bd^3}.$$

It is usually important to find the maximum stress. In this case the maximum tensile and compressive stresses are at $y = \pm\frac{d}{2}$, and are of magnitude

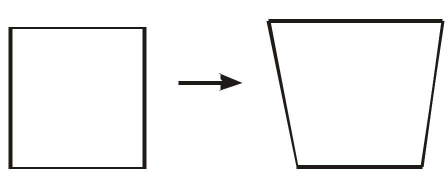
$$\sigma_{max} = \frac{3FL}{2bd^2}.$$

Deformation in bending

Whereas there is stress only along the axis (x direction) of the beam, Poisson's Ratio effects ensure that there are strains in all three directions. In particular, along the z direction, Hooke's Law gives

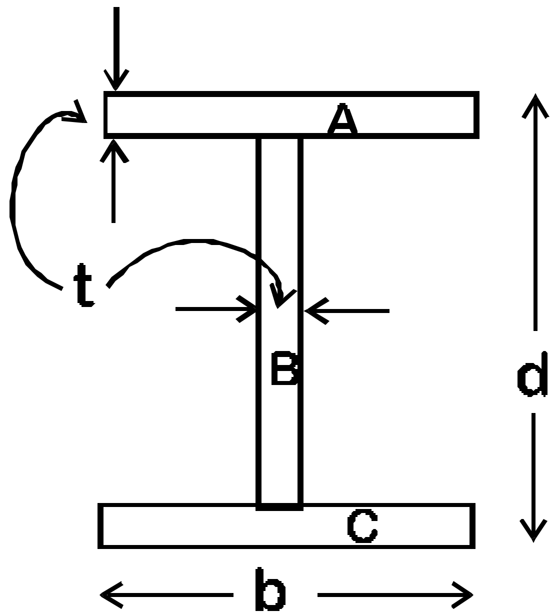


$$\varepsilon_z = \frac{1}{E}(\sigma_z - \nu(\sigma_x + \sigma_y)) = -\frac{\nu\sigma_x}{E}.$$



Thus ε_z is positive where the stress is negative, and negative where the stress is positive. The displacements along z are therefore such as to make the beam thinner along z where the bending stress is positive, and thicker where the stress is negative. For positive bending (such as the three-point bend case above, with tension along the bottom and compression along the top) the effect on an initially rectangular cross section would be to change the shape to that illustrated.

Example – I-section beam



The beam is in three-point bend, bending about a horizontal axis. We are to calculate the maximum stress and compare it with that in a square section of the same area.

We are to use the bending equation $\frac{\sigma}{y} = \frac{M}{I}$, i.e. $\sigma = \frac{My}{I}$. The maximum stress corresponds to maximum y. y is the distance from the neutral axis, which because of the symmetry is through the centre of the section; so maximum y is given by d/2. Maximum stress also corresponds to maximum M, at the centre of the beam beneath the load point; for load F and distance L between outer supports, $M = \frac{FL}{4}$. So for the I section the maximum stress is given by

$$\sigma = \frac{FLd}{8I} \qquad \text{(E1)}.$$

Now we need to calculate I. This is done by adding together the second moments of the components A, B and C. For a rectangle, the second moment about an axis of symmetry is given by the standard result "$I = \frac{bd^3}{12}$". We can use this immediately to get the second moment of the component B about the neutral axis:

$$I_B = \frac{t(d-2t)^3}{12} \qquad \text{(E2)}.$$

For the component A, its second moment about its horizontal axis is given by $\frac{bt^3}{12}$. However, we need the second moment about the axis through the centre of the beam; to get this, use the parallel axis theorem. Then,

$$\begin{aligned} I_A &= \frac{bt^3}{12} + bt\left(\frac{d}{2} - \frac{t}{2}\right)^2 \\ &= \frac{bt^3}{12} + \frac{bt(d-t)^2}{4} \end{aligned} \qquad \text{(E3)}.$$

Similarly,

$$I_C = I_A = \frac{bt^3}{12} + \frac{bt(d-t)^2}{4} \qquad \text{(E4).}$$

The total I is given by $I_A + I_B + I_C$. Using equations (E2), (E3) and (E4) this is

$$I = \frac{t(d-2t)^3}{12} + 2\left(\frac{bt^3}{12} + \frac{bt(d-t)^2}{4}\right) \qquad \text{(E5).}$$

To give a specific example, put d = b and t = b/8. Then (E5) becomes

$$I = \frac{(b/8)(3b/4)^3}{12} + 2\left(\frac{bb^3}{8^3 \times 12} + \frac{b(b/8)(7b/8)^2}{4}\right)$$

so that

$$I = \frac{b^4}{8^3}\left(\frac{27 + 2 + 6 \times 49}{12}\right) = 5.2572 \times 10^{-2} b^4 .$$

Using this figure in (E1) gives

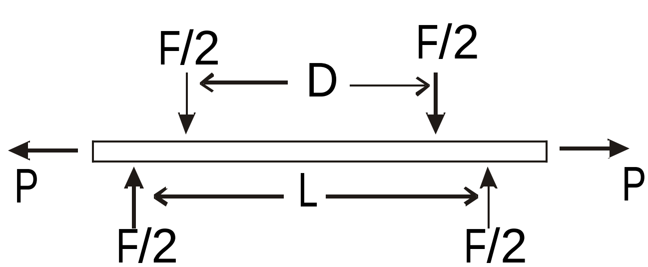
$$\sigma = 2.378 \frac{FL}{b^3} \qquad \text{(E6).}$$

The total area of the I section is given by (d-2t)t + 2bt = b(3/4)b/8 + 2b(b/8) = $.34375b^2 = (.5863b)^2$. Therefore, a square section of the same area has side .5863b. The second moment for this section is $(.5863b)^4/12$. The maximum stress is given by (equation (E1))

$$\sigma = \frac{FL(.5863b)}{8(.5863b)^4/12} = \frac{7.44FL}{b^3}$$

Comparing this result with that of equation (E6) shows that the I section is a more efficient structure than the square. Distribution of more material at the top and bottom, where stresses are greatest, leads to a lower value of maximum stress.

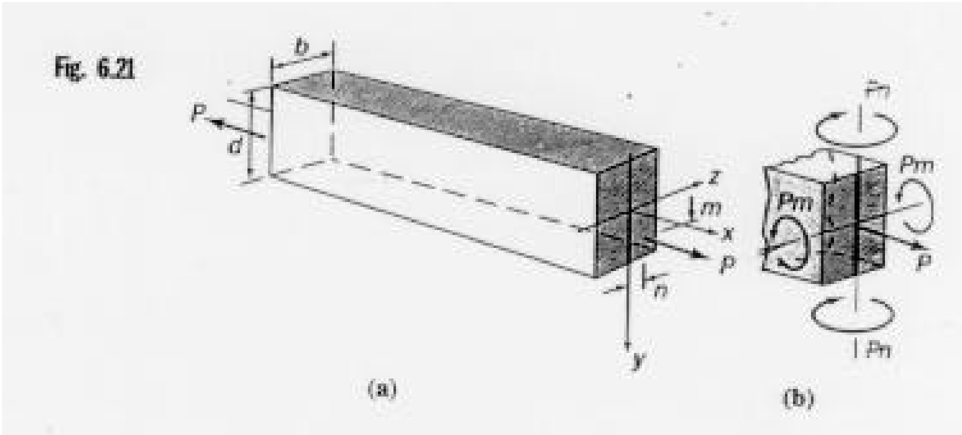
Example - superposition



What force P is required to eliminate tensile bending stresses? Assume a circular section of radius R.

Eccentric end loading

Suppose a bar is loaded axially at points on its end faces that are not at the centroid, as shown. Bending moments result. Suppose we have y and z axes passing through the centroid. Then, for the force P, distant m from the z axis, there is a moment about the



z axis of mP. This will give rise to a bending stress, given by the bending equation as $\frac{mPy}{I_z}$ at any point distant y from the z axis. Similarly, there will be other bending stresses caused by the moment nP acting about the y axis, giving a stress $\frac{nPz}{I_y}$.

Additionally, there is a direct tensile stress from the force P of P/A. By superposition, the total stress is therefore

$$\sigma_x = \frac{P}{A} + \frac{Pmy}{I_z} + \frac{Pnz}{I_y}.$$

For the rectangular cross section shown,

$I_z = \frac{bd^3}{12};\ I_y = \frac{b^3 d}{12};\ A = bd$ and we may rewrite the stress as

$$\sigma_x = \frac{P}{bd}\left[1 + \frac{12my}{d^2} + \frac{12nz}{b^2}\right].$$

Since y and z can be either positive or negative, the sign of the stress will also vary. Suppose P is negative, so that the beam is in compression; there may be areas where there are tensile stresses. In civil engineering applications, it may be important, if the beam material is brittle, to avoid tension. If there is tension, it will be a maximum at the surfaces y = -d/2 and z = -b/2. Putting these values in the equation above gives

$$\sigma_{max} = \frac{P}{bd}\left[1 - \frac{6m}{d} - \frac{6n}{b}\right].$$

For there to be no tension, $\sigma_{max} = 0$ and the above equation becomes

$$0 = \frac{P}{bd}\left[1 - \frac{6m}{d} - \frac{6n}{b}\right]$$

which implies

$$1-\frac{6m}{d}-\frac{6n}{b}=0\,.$$

This defines a condition on m and n, i.e. a restriction on the position of the force P, which must be satisfied if there is to be no tension. It defines an area of the beam end, inside which the load P must be located to avoid tension. The area is diamond-shaped, of height d/6 and semi-width b/6.